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# Effects of Illegal Gold Mining Activities on heavy Metal Content of Soil in Chanchaga, Niger State

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## Abstract
This field study was conducted to assess heavy metal concentrations in soils as a result ofillegal gold mining activities around the mining site in Chanchaga Village in Niger State, Nigeria. Gold mining and exploration has been identified as the major activity in this area. Soil samples obtained at distances of 10, 20, 30, 40, 50 and 60m from the mining site and soil outside the mining site as control were obtained, digested and analysed for the concentrations of heavy metals like zinc, lead, copper and chromium using standard methods. These toxic heavy metals were observed to be significantly present in high concentrations in the soil samples obtained from the gold mining farm site when compared to the soil sample obtained outside the mining environment and World Health Organisation maximum permissible standard. The heavy metal concentrations of zinc and copperobserved in the control sample were 3.0 and 1.42milligram per kilogram respectively and all were lower than those obtained from the soil samples at the gold mining site while leadand chromium were not detected. This implies thatillegal gold mining activities has contaminated soil in the mining site with the presence of toxic heavy metals.

**Keywords:** Gold mining, Chanchaga, Heavy metals, Soil, WHO

## INTRODUCTION

Human activities involved in the exploitation of the environment for wealth creation has resulted to environmental pollution directly or indirectly. Mineral prospecting is one activitythat can contribute to national wealth due to major roles minerals such as gold, diamond, copper; zinc, iron ore, lead and columbite play in industries and human life.For instance, the early discovery of gold in Langlaagte Farm near Johannesburg in 1886 saw the setting up in 1889 of a gold mining prospectus by freelance diggers referred to as Chamber of Mines (Chamber of Mine, 2002). In Nigeria, mineral mining is not a major source of revenue as it lost government's interest after the Second World War due to the discovery of crude oil (Godwin, 2014). Mining is a prosperous and expanding industry that is capable of providing sustainable economic, social and rural development to our communities. Despite its wealth creation prospect to the nation, it also poses danger to land, fauna, flora and water in the environment it operates (Yang, Shua, Qiu, Wang and Lang, 2004).

Mining process involvesdigging or excavation of land which can bring infertile soil from the in-depth to the land surface; this soil is not nutritious and therefore not good for agriculture. Also the excavated land when left uncovered after the mining activities renders the land useless (Rivas and Cendrero, 2006).Gold processing activities in Chanchagafor a century now involves the crushing of rocks and sediment which are mixed with water for separation of gold and the effluent left to flow freely to the ground may sink into the soil or flow into the adjourning flowing river in the region. These activities, of no doubt release heavy metals to the soil and water bodies. Heavy metal contamination of soil and water is hazardous to the ecosystem. Metals may exist in compound form binding rocks or sediment together and may be released to the environment in that form which could lead to changes in chemical contents of soils and water bodies (Sawyer, 1986). The level of contaminations depend on the concentration of the metals in the environment, but study have shown that the contaminant characteristic of an element depend on its specific binding form and its level of reactivity rather than its concentration (Sawyer, 1986). Heavy metals are mobile and distribute themselves throughout the soil and water which make them available for plant uptake, fish and animal, their mobility are influenced by some factors, these factors including soil pH, soil texture and organic matters which play major roles in the form and bio-availability of heavy metals, also the behaviours of metals are determined by a range of physico-chemical process that influenced their mobility and availability in soil (Godwin, 2014).Heavy metals such as Lead (Pb), Mercury (Hg), Arsenic (As), Chromium (Cr), Cadmium (Cd), Nickel (Ni), Cobalt (Co), and copper (Cu) are known to have negative effect on soil, and as a result of this, crop yield is drastically reduced.

The types of heavy metal present depend on the type of chemical binding the rocks or sediments, hence, to understand the potential effect of heavy metal and their complexity in ecological system it is important to study the various forms of the elements present in the rocks or sediments in the area (Godwin, 2014). Many cases of heavy metals toxicity have been reported worldwide and the effects on human health have been regularly reviewed by International bodies like World Health Organization (WHO) (Nwajei and Gagophien, 2000). Similar threat has been reported in Nigeria from lead mining area in the village ofDareta in Zamfara State where children were reported with lead concentrationof 119.4µg/dl in their blood that causes neurological disorder. The limit of lead concentration in America and France are 400ppm while more than 100,000ppm were found around this affected village in Nigeria (Galadima and Garba, 2012).Heavy metal contamination of soil can make the soil infertile and therefore limit agricultural activities in such environment which inhibit large farm production. Agricultural production on large scale can be inhibited by deficiency in essential element such as manganese, zinc or copper which are micronutrient to plants, or cobalt, zinc and copper for livestock which require in small quantity. Some heavy metals such as arsenic, mercury, cadmium, lead and nickel are harmful to crop even at low concentration (Mandor, 2002).

## MATERIALS AND METHODS

### The study site

The gold mining site is located in Chanchaga, Chanchaga Local Government Area of Niger state. It is located on Latitude 9'.53"33° and Longitude 6'.58"33°, it is 21 km from Paiko and 20.9 km from Tungan in Niger state. The area just like many other places in Niger state has mean annual rainfall of 1334 mm starting from April and last till October.The physical conditions of Chanchaga significantly influence the economic activities of the people; the activities of the mining companies have also influenced the economic life of the people. The principal occupation of the people is agriculture, such as farming and gold mining.

### Collection of Soil samples

**Soil samples were obtained from farmlands in the study area at distances 10, 20, 30, 40, 50 and 60m from the gold mining site for analyses to determine the effects of the gold mining on soilmetal contents.** Soil collected from a farm plot in Kpakungu area, Niger State, where mineral processing activities does not take place served as the control sample.Each soil sample obtained was properly packaged and labelled in a brown envelope and sent to the laboratory for analysis.

### Preparation of samples for analysis

The labelled soil samples collected were analysed at theSoil Science Department laboratory, School of Agriculture, Federal University of Technology, Minna (FUT). Soils were oven-dried for 2 hours to remove residual moisture, ground using mortar and pestle and sieved using 0.5 mm mesh sizessieve and made ready for digestion. For characterization analysis, the samples were sieved using 0.2mm mesh size, kept in a polythene bags and labelled accordingly ready for analysis (Udo and Ogunwale,1986).

### Metal Determination using AAS

2g of prepared soil sample were weighed using electronic weighing balance and transferred into 100 ml flask; about 80.0 ml of a mixture of (60.0 ml HCl and 20.0 ml $HNO_3$) were added to each samples and placed onto the hot plate. The samples were heated at 200$^{o}$C until a light colour solution was obtained (Godwin, 2014), and the samples were then allowed to cool, filtered and the filtrate diluted to the mark with de-ionized water, mixed thoroughly and then tested for heavy metals using Atomic Absorption Spectrophotometer (AAS). The following metals(Pb, Cu, Cr and Zn) were analyzed using Atomic Absorption Spectrophotometer and potassium and sodium using flame photometer.The characterizations of the samples were done according to the procedures reported by Njosi (2010). Different extracting solutions were added and the mixture shaken thoroughly, after the extraction, the solutions were filtered and the filtrate used for determiningheavy metal contents.In the AAS, different heavy metals absorb different light with specific frequency and excite their electron from their ground energy level to a higher energy level after being vaporized. The corresponding metal lamp used emits light with the required frequency which is absorbed by the metal ion in the sample. The amount of light absorbed by metal ion in the sample is proportional to the concentration of the metal ion in the solution.

**RESULTS AND DISCUSSION**

The metal contents in the soil samples obtained at different distances from the gold mining site are presented in Table 1.

Table 1: Metal Concentration in Soil samples obtained from the study area at different distances

| Samples | Distances (m) | K mg/kg | Ca mg/kg | Mg mg/kg | Na mg/kg | Zn mg/kg | Pb mg/kg | Cu mg/kg | Cr mg/kg |
|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 10 | 5.75 | 11.20 | 22.01 | 13.82 | 17.55 | 2.63 | 4.36 | 3.99 |
| $S_2$ | 20 | 6.86 | 13.41 | 27.10 | 13.54 | 14.31 | 1.81 | 2.83 | 1.80 |
| $S_3$ | 30 | 7.93 | 11.08 | 30.56 | 12.44 | 13.16 | 0.93 | 2.93 | 1.4 |
| $S_4$ | 40 | 21.92 | 10.86 | 38.40 | 13.86 | 16.25 | 1.17 | 3.01 | 1.82 |
| $S_5$ | 50 | 27.55 | 10.56 | 47.60 | 14.32 | 10.77 | 0.75 | 2.51 | 0.93 |
| $S_6$ | 60 | 30.84 | 18.32 | 53.10 | 12.96 | 11.43 | 0.92 | 2.90 | 1.09 |
| $S_c$ | Control | 35.83 | 47.87 | 58.06 | 26.78 | 3.00 | ND | 1.42 | ND |
| W.H.O | | - | - | - | - | 3.0 | 0.01 | 1.0 | 2.0 |

The potassium concentrations of the soil samplesfrom the farm around the gold mining siteat distances 10, 20, 30, 40, 50 and 60m as presented in table 1 were 5.75, 6.86, 7.93, 21.92, 27.55 and 30.84 mg/kg respectively, and lower when compared to 35.83 mg/kg obtained from the control sample. This shows an increasing trend in the potassium contents of the soil samples as the distances from the farm to the gold mining siteincreases indicating that the mining activities do not affect K concentration. Ca, Mg and Na exhibited similar pattern of the soil samples at these distances. Heavy metals like zinc, lead, copper and chromium considered in this study have been discovered to be released to the environment by human activities, the source of thesemetal pollutants are mining, metal smelting, metallurgical industries and other mineral processing industries, metal corrosion, waste disposal, agriculture and forestry, fossil fuel combustion, sport and leisure activities, the world industrialization generally increases heavy metal contamination (Mandor, 2002).These heavy metals can make the soil infertile when present at high concentrations because of their toxicity andcan inhibit large farm production.This is in agreement with this study as the heavy metals analysed as presented in table 1 were observed to be higherin the soil samples obtained at the mining site than the soil sample obtained outside the mining environment. This shows the effect of illegal gold mining on agricultural soils. According to Mishra and Singh (1987), it was found that the contamination of soil with illegal gold mining activities leads to an increase in the zinc, lead, copper and chromium contents of the soil. Some heavy metals such as arsenic, mercury, cadmium, lead and nickel are harmful to crops even at low concentration (Mandor, 2002).This is evident in this study as the soil samples obtained from the gold mining farm site had high concentrations of these heavy metals tested for when compared to the soil sample outside the gold mining environment (control).

Heavy metals tested for in the soil samples were zinc, lead, copper and chromium. Zinc concentrations in soil samples obtained at distances from the gold mining farm site were in a decreasing order and still higher than 3.0 mg/kg recorded in the control sample. According to Tobias *et. al.* (2013), World Health Organization's maximum permissible limit of zinc concentration in agricultural soil is 3.0 mg/kg which is less than zinc concentrations in soil samples from the gold mining environment. For lead concentrations in the soil samples at these distances also decreased as the distances increased, though they remained higher than the World Health Organization's maximum permissible limit of 0.01mg/kg in agricultural soil as reported by Tobias et. al. (2013), indicating lead contamination in the soil. Lead in the control soil sample obtained outside the gold mining site was undetected. Also, copper concentrations in the soil samples from studied sites at the designated distances also followed a decreasing trend but was however, higher than 1.42 mg/kg of the control sample and World Health Organization's maximum permissible limit of 1.0 mg/kg in agricultural soil as reported by Tobias et. al. (2013). The concentrations of copper in soil samples were slightly above the World Health Organization's maximum permissible limit and this agrees with the report of *Sharon, et. al. (2013), who reported that copper* is a very common substanceoccurring naturally in the environment and spreading through the environment through natural phenomena. Chromium, one of the metals tested for in the soil samples at various distances exhibited decreased concentrations as the distances increased. These concentrations in the soil samples were within World Health Organization's maximum permissible limit of 2.0 mg/kg in agricultural soil as reported by Tobias, *et. al.*, (2013), except soil sample obtained at 10m away from the gold mining site. Chromium concentration at this distance was slightly higher than the World Health Organization's maximum permissible limit. The control soil sample had no detectable chromium concentration. The presence of high concentrations of zinc, copper, lead and chromium metals in soil samples obtained near the gold mining site is an indication of soil contamination arising from the various gold mining activities in the environment. High soil metal contents as revealed in this study for some areas have serious public health implication for the residents of this community, given the catalogue of health effects associated with exposure to metals generally, and toxic heavy metals in particular.Ingeneral, heavy metals are systemic toxins with specificneurotoxic, nephrotoxic, fetotoxic and teratogenic effects.Heavy metal toxicity can directly influence behaviour by impairingmental and neurological functions, influencing neurotransmitterproduction and utilization, and altering numerousmetabolic body processes. Systems in which toxic metalelements can induce impairment and dysfunction includethe blood, cardiovascular, eliminative pathways (colon,liver, kidneys, skin), endocrine (hormonal), energy productionpathways, enzymatic, gastrointestinal, immune,nervous (central and peripheral), reproductive andurinary systems. These toxic impacts of metals areaffected through the formation of free radicals (highlyreactive oxygen species), a disturbance of the pro andantioxidant balance of the body, both of which can causecell damage by destruction of proteins, degradation of nucleic acids or lipid peroxidation (*Sharon*,et.al., 2013). It was further reported that toxic heavy metals have beenimplicated in many health defects arising due to theirtoxic effects involving several body tissues, organs and systems (Godwin 2014).

## Conclusion

The characterization of the soil samples obtained at varying distances from the gold mining site were observed to be high in toxic heavy metals (Zn, Pb, Cu, Cr). These observations are indications that soil samples at the gold mining farm site are contaminated and unsuitable for plant growth as negative health implications may be incurred if such plants are consumed. Obviously, animals that graze on plants on this soil are liable to be affected with health problems as a result of the contaminated soil with heavy metals.

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# On the Auxiliary Lyapunov Functions and Uniform Practical Stability for Nonlinear Impulsive Caputo Fractional Order Derivative via New Modelled Generalized Dini Derivative

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**Abstract.**

In this paper, the uniform practical stability of the trivial solution of a nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse is examined using an auxilliary Lyapunov function which are analogues of vector Lyapunov functions, and is generalized by a class of piecewise continuous Lyapunov functions. Together with comparison results, sufficient conditions for the uniform practical stability for impulsive Caputo fractional order systems are established. An illustrative example is given to confirm the suitability of the obtained results.

**Keywords:** Uniform practical stability, Caputo derivative, impulse, Lyapunov function.

MSC: 34A08; 34A34; 34A37; 34D20.

## 1 INTRODUCTION

Spanning three centuries, the history of fractional calculus has been marked by significant developments, largely confined to the theoretical foundations of mathematics. Moreover, recently, its relevance have also span into several other fields like the mathematical physics, engineering, control theory, economics, etc. The earliest – more or less systematic studies of the concept seem to have been made in the $19^{th}$ century by Liouville, Riemann, Caputo, etc. [37], and in the last four decades, it was observed that the advancement of fractional calculus has enabled the description of complex system behaviors through fractional differential systems (including many physical phenomenon having memory and genetic characteristics), thus providing new insights into their dynamics.

As observed in [6, 27], one of the trends in the stability theory of solutions of differential equations is the so-called practical stability. This aspect of stability was introduced by LaSalle and Lefschetz in [24], and it is used in estimating the worst-case transient and steady-state responses together with verifying pointwise in time constraints imposed on the trajectory curve. Fundamental results in this area were established in [9, 31] and [40] for integer order derivative.

In line with the development of the theory of practical stability in recent years is the mathematical theory of impulsive differential equations which have experienced a massive research attention and development. Now, the theory of impulsive differential equations is richer than the corresponding theory of differential equations [16] as they constitute very important models for describing the true state of several real life processes and phenomena, since many evolution processes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly (see [16]).

Moreover, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [35], and one of the most versatile methods in the study of the stability properties of impulsive systems is the Lyapunov's direct method also called the Lyapunov's second method. Interestingly, the method involves seeking an appropriate continuously differentiable function (called Lyapunov's function) that is positive definite and whose time derivative along the solution path or trajectory curve is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [12] and [30]. Investigations into the stability of fractional order systems is a recent development and major obstacles in utilizing the Lyapunov's functions for these systems lies in establishing an appropriate definition of the derivative amidst fractional differential equations. However, this difficulty was safely handled in [1] where the choice of a Lyapunov function that is continuous was adopted, with a re-definition, as it were, of the Dini derivative for the given fractional order system with respect to the Lyapunov function. To the best of our knowledge, not so much have been done in the literature in terms of practical stability for impulsive fractional order derivative using the vector Lyapunov functions by means of the comparison method.

Against this background, [6] established sufficient conditions for the practical stability of impulsive fractional order systems using the vector Lyapunov functions with the comparison method, while fundamental results for the stability of impulsive Caputo fractional differential equations using the scalar Lyapunov function were examined in [1] and [2]. Other stability results have been established in [42-48].

In this paper, the uniform practical stability of impulsive Caputo fractional order systems is examined. By means of the comparison principle, sufficient conditions for the uniform practical stability of impulsive fractional order systems is established using a class of piecewise continuous functions. An illustrative example given, extends and improves on existing results.

## 2 Preliminary Notes and Basic Definitions

Fractional calculus is considered as a natural generalization of the classical calculus of integer order to arbitrary order, thereby allowing for the extension of the traditional concepts of derivative and integral to functions with fractional orders. By this extension, functions with non integer orders are much more flexible in describing real world systems (see [14, 25, 26, 34 and 37]).

There are several definitions of fractional derivatives and fractional integrals.

**General case.** Let the number $n-1<\beta<n, \beta>0$ be given, where $n$ is a natural number, and $\Gamma(.)$ denotes the Gamma function.

**Definition 2.1.**

The Riemann Liouville fractional derivative of order $\beta$ of $\gamma(t)$ is given by (see [35])

$$ {}^{RL}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t\geq t_0 $$

**Definition 2.2.**

The Caputo fractional derivative of order $\beta$ of $\gamma(t)$ is given by (see [35])

$$ {}^{C}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t\geq t_0 $$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

**Definition 2.3.**

The Grunwald-Letnikov fractional derivative of order $\beta$ of $\gamma(t)$ is given by (See [1])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0 $$

and

**Definition 2.4.** The Grunwald-Letnikov fractional Dini derivative of order $\beta$ of $\gamma(t)$ is given by (See [1])

$$^{GL}D_0^{\beta}\gamma(t) = \lim_{h\to 0^+}\sup \frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r\,\beta}C_r\gamma(t-rh),\ t\ge t_0$$

where $^{\beta}C_r$ are the binomial coefficients and $\left[\dfrac{t-t_0}{h}\right]$ denotes the integer part of $\dfrac{t-t_0}{h}$.

**Particular case** (when n=1). In most applications, the order of $\beta$ is often less than 1, so that $\beta\in(0,1)$. For simplicity of notation, we will use $^{C}D^{\beta}$ instead of $^{C}_{t_0}D^{\beta}$ and the Caputo fractional derivative of order $\beta$ of the function $\gamma(t)$ is

$$^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{-\beta}\gamma'\,ds,\ \ t\ge t_0$$

(2.1)

## 3 Impulses in Fractional Dynamics

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0<\beta<1$.

$$^{C}D^{\beta}\gamma(t) = f(t,\gamma),\ t\ge t_0,$$
$$\gamma(t_0)=\gamma_0,$$

(3.1)

where $\gamma\in R^N$, $f\in C[R_+\times R^N, R^N]$, $f(t,0)\equiv 0$ *and* $(t_0,x_0)\in R_+\times R^N$.

Some sufficient conditions for the existence of the global solutions to (3.1) are considered in [7], [11], [22], [23], [29], [35], [42].

The IVP for FrDE (3.1) is equivalent to the following Volterra integral equation (See [2]),

$$\gamma(t)=\gamma_0+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\gamma(s))ds,\ t\ge t_0$$

(3.2)

Consider the IVP for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0<\beta<1$,

$$
\begin{gathered}
{}^{C}D^{\beta}\gamma(t) = f(t,\gamma),\ t \geq t_0,\ t \neq t_k,\ k = 1,2,\ldots \\
\Delta\gamma = I_k(\gamma(t_k)),\ t = t_k,\ k \in N, \\
\gamma(t_0^+) = \gamma_0,
\end{gathered}
\tag{3.3}
$$

where $\gamma, \gamma_0 \in R^N$, $f \in C[R_+ \times R^N, R^N],\ and\ t_0 \in R_+\ I_k : R^N \to R^N,\ k = 1,2,\ldots.$

under the following assumptions:

$(i)\, 0 < t_1 < t_2 < \ldots < t_k < \ldots,\ and\ t_k \to \infty\ as\, k \to \infty;$

$(ii)\, f : R_+ \times R^N \to R^N$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $x \in R^N, k = 1,2,\ldots,$ and $\lim\limits_{(t,y)\to(t_k^+,\gamma)} f(t,y) = f(t_k^+,\gamma)$ exists;

$(iii)\, I_k \times R^N \to R^N$

In this paper, we assume that $f(t,0) \equiv 0,\ I_k(0) \equiv 0$ for all $k$ so that we have trivial solution for (3.3), and the points $t_k,\ k = 1,2,\ldots$ are fixed such that $t_1 < t_2 < \ldots.\ and\ \lim\limits_{k\to\infty} t_k = \infty.$ The system (3.3) with initial condition $\gamma(t_0) = \gamma_0$ is assumed to have a solution $\gamma(t; t_0, \gamma_0) \in PC^{\beta}([t_0, \infty), R^N).$

**Remark 3.1.** The second equation in (3.3) is called the impulsive condition, and the function $I_k(\gamma(t_k))$ gives the amount of jump of the solution at the point $t_k$.

**Definition 3.1.** Let $\Omega : R_+ \times R^N \to R^N$ Then $\Omega$ is said to belong to class $\chi$ if,

(i) $\Omega$ is continuous in $(t_{k-1}, t_k]$ and for each $\gamma \in R^N$ and $\lim\limits_{(t,y)\to(t_k^+,\gamma)} \Omega(t,y) = \Omega(t_k^+,\gamma)$ exists;

(ii) $\Omega$ is locally Lipschitz with respect to its second argument $x$ and $\Omega(t,0) \equiv 0$

Now, for any function $\Omega(t,\gamma) \in PC([t_0,\infty) \times \xi, R_+^N)$. we define the Caputo fractional Dini derivative as:

$$
{}^{C}D_+^{\beta}\Omega(t,\gamma) = \lim_{h\to 0^+} \sup \frac{1}{h^{\beta}} \{\Omega(t,\gamma) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [\Omega(t - rh, \gamma - h^{\beta} f(t,\gamma) - \Omega(t_0,\gamma_0)]\}
\tag{3.4}
$$

$t \geq t_0$ where $t \in [t_0, \infty),\ \gamma, \gamma_0 \in \xi,\ \xi \in R^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T].$

**Definition 3.2.** A function $g \in PC[R^n, R^n]$ is said to be quasimonotone nondecreasing in $\gamma$, if $\gamma \le y$ and $\gamma_i = y_i$ for $1 \le i \le n$ implies $g_i(\gamma) = g_i(y)$.

**Definition 3.3.** The zero solution of (3.3) is said to be:

(*PS1*) practically stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ continuous in $t_0$ such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \le \delta$ implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \ge t_0$;

(*PS2*) uniformly practically stable if for every $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$, continuous in $t_0$ such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \le \delta$ implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \ge t_0$;

(*PS3*) asymptotically practically stable if it is practically stable and if for each $\varepsilon > 0$ *and* $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \ge t_0 + T$ and $\|\gamma_0\| \le \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$;

(*PS4*) uniformly asymptotically practically stable if it is uniformly practically stable and $\delta_0 = \delta_0(\in)$ and $T = T(\varepsilon)$ such that for $t \ge t_0 + T$, the inequality $\|\gamma_0\| \le \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

**Definition 3.4.** A function $a(r)$ is said to belong to the class $K$ if $a \in PC([0, \psi), R_+)$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing in $r$.

In this paper, we define the following sets:

$$\overline{S}_\psi = \{\gamma \in R^N : \|\gamma\| \le \psi\}$$

$$S_\psi = \{\gamma \in R^N : \|\gamma\| < \psi\}$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{gathered} {}_{t_0}^{C}D^{\beta} u = g(t, u),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,... \\ \Delta u = \psi_k(u(t_k)),\ t = t_k,\ k \in N, \\ u(t_0^+) = u_0, \end{gathered} \tag{3.5}$$

existing for $t \geq t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g : R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where $g$ is the continuous mapping of $R_+ \times R_+^n$ into $R^n$. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (3.5) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC^{\beta}([t_0, \infty), R^n)$.

**Lemma 3.2.** Assume $m \in PC([t_0, T] \times \overline{S}_{\psi}, R^N)$. and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of $m$ exists at $t^*$, then the inequality ${}^C D_+^{\beta} m(t^*, \alpha_1) - {}^C D_+^{\beta} m(t^*, \alpha_2) > 0$ holds.

***Proof.*** Let $\Omega(t, \gamma) = m(t, \alpha_1) - m(t, \alpha_2)$. Applying (3.4), we have

$$\begin{aligned} {}^C D_+^{\beta}(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \{ &[m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\ &- \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \} \end{aligned}$$

when $\alpha_1 = \alpha_2$ we have

$$\begin{aligned} {}^C D_+^{\beta}(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \{ &-[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\ &- \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \} \end{aligned}$$

$$\begin{aligned} {}^C D_+^{\beta}(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = &-\limsup_{h \to 0^+} \frac{1}{h^{\beta}} [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\ &+ \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\ &- \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \end{aligned}$$

$$\begin{aligned} {}^C D_+^{\beta}(m(t^*, \alpha_1) - m(t^*, \alpha_2)) = &-\limsup_{h \to 0^+} \frac{1}{h^{\beta}} [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\ &- \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \end{aligned}$$

$$ {}^{C}D_{+}^{\beta}(m(t^*,\alpha_1)-m(t^*,\alpha_2)) = -\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)] $$

Applying equation (3.8) in [1], we have

$$ {}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) = -\frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)}\left[m(t^*,\alpha_1)-m(t^*,\alpha_2)\right] $$

By the lemma, we have that

$$ m(t,\alpha_1)-m(t,\alpha_2)<0 \quad for\ t_0 \le t \le t^* $$

And so, it follows that

$$ {}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) > 0 $$

Note that some existence results for (3.5) are given in [11, 13] and [14].

**Remark 3.3.** Lemma 3.2 extends Lemma 1 in [1], where the vectors $m(t,\alpha_1)$ *and* $m(t,\alpha_2)$ are compared component-wise.

In the following, we establish the comparison result for the system (3.3).

## 4 Fractional Differential Inequalities and Comparison Results for Vector Fractional Differential Equation

In this section, again we assume that $0<\beta<1$.

**Theorem 4.1. (Comparison Result)**

Assume that

i) $g \in PC[R_+ \times R_+^n, R^n]$ and is continuous in $(t_{k-1}, t_k]$, $k = 1,2,...$ and $g(t,u)$ is quasimonotone non-decreasing in $u$ for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$$ {}^{C}D_{+}^{\beta}\Omega(t,\gamma) \le g(t,\Omega(t,\gamma)),\ \ (t,\gamma)\in R_+ \times R^N $$

and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t, \gamma(t))),\ t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iii) $\vartheta(t) = \vartheta(t, t_0, u_0) \in PC^\beta([t_0, T], R^n)$ be the maximal solution of the IVP for the IFrDE system (3.5)

Then,

$$\Omega(t, \gamma(t)) \le \vartheta(t),\ \ t \ge t_0, \tag{4.1}$$

where $\gamma(t) = \gamma(t, t_0, \gamma_0) \in PC^\beta([t_0, T], R^N)$ is any solution of (3.3) existing on $[t_0, \infty)$, provided that

$$\Omega(t_0^+, \gamma_0) \le u_0 \tag{4.2}$$

**Proof.**

Let $\eta \in S_\psi$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations,

$$\begin{aligned} {}^C D^\beta u &= g(t,u) + \eta, \quad for\ t \in [t_0, \infty) \\ u(t_0^+) &= u_0 + \eta, \end{aligned} \tag{4.3}$$

for $t \in [t_0, \infty)$.

The function $u_\eta(t, \alpha)$ is a solution of (4.3), where $\alpha > 0$ the fractional differential equation (3.5) if and only if it satisfies the Volterra fractional integral equation,

$$u_\eta(t, \alpha) = u_0 + \eta + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} (g(s, u_\eta(s, \alpha)) + \alpha) ds,\ \ t \in [t_0, \infty). \tag{4.4}$$

Let the function $m(t, \alpha) \in PC([t_0, T] \times S_\psi, R_+^N)$ be defined as $m(t, \alpha) = \Omega(t, \gamma^*(t))$

We now prove that

$$m(t, \alpha) < u_\eta(t, \alpha) \quad for\ \ t \in [t_0, \infty) \tag{4.5}$$

Observe that the inequality (4.5) holds whenever $t = t_0$ *i.e.*

$$m(t_0, \alpha) = \Omega(t_0, \gamma_0) \le u_0 < u_\eta(t_0, \alpha)$$

Assume that the inequality (4.5) is not true, then there exists a point $t_1 > t_0$ such that

$$m(t_1, \alpha) = u_\eta(t_1, \alpha) \ \ and \ \ m(t, \alpha) < u_\eta(t, \alpha) \text{ for } t \in [t_0, t_1).$$

It follows from Lemma 3.2 that

$${}^C D_+^\beta (m(t_1, \alpha) - u_\eta(t_1, \alpha)) > 0 \ \ i.e.$$

$${}^C D_+^\beta (m(t_1, \alpha)) > {}^C D_+^\beta u_\eta(t_1, \alpha)$$

$${}^C D_+^\beta \Omega(t_1, \gamma(t_1)) > {}^C D_+^\beta u_\eta(t_1, \alpha)$$

and using (4.3) we arrive at

$${}^C D_+^\beta \Omega(t_1, \gamma(t_1)) > g(t_1, u(t_1, \alpha)) + \eta > g(t_1, u(t_1, \alpha)$$

Therefore,

$${}^C D_+^\beta m(t_1, \alpha) > g(t_1, u(t_1, \alpha)) \tag{4.6}$$

From Theorem 4.1, the function $\gamma^*(t) = \gamma(t, t_0, \gamma_0)$ satisfies the IVP (4.3) and the equality

$$\limsup_{h \to 0^+} \tfrac{1}{h^\beta}[\gamma^*(t) - \gamma_0 - S(\gamma^*(t), h)] = f(t, \gamma^*(t)), \ \ holds \tag{4.7}$$

where $\gamma^*(t) = \gamma(t, t_0, \gamma_0)$ is any other solution of (3.5), and

$$S(\gamma^*(t), h) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} {}^\beta C_r \left[\gamma^*(t - rh) - \gamma_0\right] \tag{4.8}$$

is the Grunwald Letnikov fractional derivative

Multiply equation (4.7) through by $h^\beta$

$$\limsup_{h \to 0^+} [\gamma^*(t) - \gamma_0 - S(\gamma^*(t), h)] = h^\beta f(t, \gamma^*(t))$$

$$\gamma^*(t)-\gamma_0-[S(\gamma^*(t),h)+\rho(h^\beta)]=h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t)-h^\beta f(t,\gamma^*(t))=[S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)]$$

(4.9)

Then for $t\in[t_0,\infty),$ we have

$$m(t,\alpha)-m(t_0,\alpha)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^\beta C_r\left[m(t-rh,\alpha)-m(t_0,\alpha)\right]=$$

$$\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^\beta C_r\left[\Omega(t-rh,\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\}$$

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^\beta C_r\left[\Omega(t-rh)-\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\}+$$

$$\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^\beta C_r\{[\Omega(t-rh),S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\Omega(t_0,\gamma_0)]-[\Omega(t-rh,\gamma^*(t-rh))-\Omega(t_0,\gamma_0)]\}$$

(4.10)

Since $\Omega(t,\gamma)$ is locally Lipschitzian with respect to the second variable, we have that,

$$\leq L\left|(-1)^{r+1}\right|\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)[S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\gamma^*(t-rh)]\right\|$$

where $L>0$ is a Lipschitz constant.

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)[S(\gamma^*(t),h)+\rho(h^\beta))-(\gamma^*(t-rh)-\gamma_0)\right\|$$

(4.11)

Using equation (4.8), equation (4.11) becomes,

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^\beta C_r)[(\gamma^*(t-rh)-\gamma_0)+\rho(h^\beta))-(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)(-1)^{r+1}[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r[(\gamma^*(t-rh)-\gamma_0)]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r\rho(h^\beta)-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L(-1)^{r+1}\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[(\gamma^*(t-rh)-\gamma_0)]+[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r\rho(h^{\beta})\right\|$$

(4.12)

Substituting equation (4.12) into (4.10) we have

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$
$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\rho(h^{\beta})\right\|$$

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$
$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\rho(h^{\beta})\right\|$$

Dividing through by $h^{\beta}>0$ and taking the $\limsup$ $as\ h\to 0^{+}$ we have,

$${}^{C}D_{+}^{\beta}m(t,\alpha)=\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\left\{\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\right\}+$$
$$\limsup_{h\to 0^+}\frac{1}{h^{\beta}}L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}{}^{\beta}C_r\rho(h^{\beta})\right\|$$

Recall that,

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)=-1\ and\ \limsup_{h\to 0^+}\frac{1}{h^{\beta}}\rho(h^{\beta})=0$$

From equations (3.6) and (3.7) in [1], we have that

$${}^{C}D_{+}^{\beta}m(t,\alpha)={}^{C}D_{+}^{\beta}\Omega(t,\gamma^*(t))+L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-(-1)-1]+0\right\|$$

Using condition (ii) of the Theorem 4.1, we obtain the estimate

$${}^{C}D_{+}^{\beta}m(t,\alpha) \leq g(t,\Omega(t,\gamma^{*}(t)) = g(t,m(t,\alpha)),$$
(4.13)

Also,

$$m(t_0^{+},\alpha) \leq u_0 \ \ and \ \ m(t_k^{+},\alpha) = \Omega(t_k^{+},\gamma(t_k) + I_k(\gamma(t_k)) \leq \rho_k(m(t_k))$$
(4.14)

Now equation (4.14) with $t = t_1$ contradicts (4.6), hence (4.5) is true.

□

For $t \in [t_0, T],$ we now establish that

$$u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \quad \text{whenever } \eta_1 < \eta_2$$
(4.15)

Observe that the inequality (4.15) holds for $t = t_0$

Assume that (4.15) is not true. Then there exists a point $t_1$ such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_2,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \ \ for \ t \in [t_0, t_1).$

By Lemma 3.2, we have that

$${}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) > 0$$

However,

$$\begin{aligned} {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) &= {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) \\ &= g(t_1,u(t_1,\alpha)) + \eta_1 - [g(t_1,u(t_1,\alpha)) + \eta_2] \\ &= \eta_1 - \eta_2 < \mathbf{0} \end{aligned}$$

which is a contradiction, and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_{\eta}(t,\alpha)\}$, $t \in [t_0, T]$ of (4.3) is uniformly bounded, i.e. there exists $\lambda > 0$ with $|u_{\eta}(t,\alpha)| \leq \lambda$ , with bound $\lambda$ on $[t_0, T]$. We now show that the family $\{u_{\eta}(t,\alpha)\}$ is equicontinuous on $[t_0, T]$. Let $k = \sup\{|\ g(t,\gamma)\ | : (t,\gamma) \in [t_0, T] \times [-\lambda, \lambda]\}$, where $\lambda$ is the bound on the family $\{u_{\eta}(t,\alpha)\}$. Fix a decreasing sequence $\{\eta_i\}_{i=0}^{\infty}(t)$, such that $\lim_{i\to\infty} \eta_i = 0$ and consider a sequence of functions $\{u_\eta(t,\alpha)\}$. Again, let $t_1, t_2 \in [t_0, T]$, *with* $t_1 < t_2$, then we have the following estimates,

$$\left\|u_\eta(t_2,\alpha)-u_\eta(t_1,\alpha)\right\| \le \| u_0+\eta_i+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}(t_2-s)^{\beta-1}(g(s,u(s,\eta_i))+\eta_i)ds$$

$$-\left(u_0+\eta_i+\frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_1-s)^{\beta-1}(g(s,u(s,\eta_i))+\eta_i)ds\right)\|$$

$$\le \frac{1}{\Gamma(\beta)}\left\|\left[\int_{t_0}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right](g(s,u(s,\eta_i))u(t_2,\eta_i)ds\right\|$$

$$\le \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right)ds\right\|$$

$$\le \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_1}(t_2-s)^{\beta-1}+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right)ds\right\|$$

$$\le \frac{k}{\Gamma(\beta)}\left\|\int_{t_0}^{t_1}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\right\|$$

$$\le \frac{k}{\Gamma(\beta)}\left[\left\|\int_{t_0}^{t_1}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}ds\right\|+\left\|\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\right\|\right]$$

$$\le \frac{k}{\beta\Gamma(\beta)}\left[\left\|(t_2-t_0)^\beta-(t_1-t_0)^\beta-(t_2-t_1)^\beta\right\|+\left\|(t_2-t_1)^\beta\right\|\right]$$

$$\le \frac{k}{\beta\Gamma(\beta)}\left[(t_2-t_1)^\beta+(t_2-t_1)^\beta\right]=\frac{2k}{\Gamma(\beta+1)}(t_2-t_1)^\beta$$

provided $\|t_2-t_1\| < \delta(\varepsilon)=\left(\frac{\Gamma(\beta+1)\varepsilon}{2k}\right)^{\frac{1}{\beta}}$, proving that the family of solutions $\{u_\eta(t,\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t,\alpha)\}$ guarantees the existence of a subsequence $\{u_{\eta_{ij}}(t,\alpha)\}$ which converges uniformly to the function $\vartheta(t)$ on $[t_0,T]$. Then we show that $\vartheta(t)$ is a solution of (4.4). Thus, equation (4.4) becomes

$$u_{\eta_{i_j}}(t,\alpha) = u_{0i_j} + \eta_{i_j} + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g_{i_j}(s,u_{\eta_{i_j}}(s,\alpha)) + \eta_{i_j})ds,\ \ t\in[t_0,\infty).$$

(4.16)

Taking the $\lim\ as\ i_j \to \infty$ in (4.16) yields,

$$\vartheta(t) = u_0 + \eta_{i_j} + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,\vartheta(t))ds,\ \ t\in[t_0,\infty).$$

(4.17)

Hence, $\vartheta(t)$ is a solution of (3.5) on $[t_0,T]$. We claim that $\vartheta(t)$ is the maximal solution of (3.5). Then, from (4.5), we have that $m(t,\alpha) < u_\eta(t,\alpha) \le \vartheta(t)\ on\ [t_0,T]$..

Suppose that in Theorem 4.1, $g(t,u) \equiv 0,$ then we have the following results

**Corollary 4.1.**

Assume that Condition (i) of Theorem 4.1 holds and,

(i) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$${}^{C}D_{+}^{\beta}\Omega(t,\gamma) \le 0$$

(4.14)

holds, and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

Then for $t \in [t_0,\infty),$ the inequality

$$\Omega(t,\gamma(t)) \le \Omega(t_0^+,\gamma_0) \text{ holds}$$

## 5 Main Results

In this section, we will obtain sufficient conditions for the uniform practical stability of the system (3.3). Again we assume $0 < \beta < 1$.

**Theorem 5.1.** Assume that

i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n$, $k = 1,2,...,$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,y) = g(t_k^+, u)$ exists, and $g(t,u)$ is quasimonotone nondecreasing in $u$

ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$$^C D_+^{\beta}\Omega(t,\gamma) \le g(t,\Omega(t,\gamma)),\ t \ne t_k,\ holds\ \ for\ all\ (t,\gamma) \in R_+ \times S_\Psi,$$

there exists a $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))),\ t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iii) $b(\|\gamma\|) \le \Omega_0(t,\gamma) \le a(\|\gamma\|)$, where $a, b \in K$ and $\Omega_0(t,\gamma), = \sum_{i=1}^{N} \Omega_i(t,\gamma).$

Then the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

***Proof.*** Let $0 < \varepsilon < \psi$ and $t_0 \in R_+$ be given.

Assume that the solution (3.5) is uniformly practically stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{N} u_{i0} < \delta \ \text{implies}\ \sum_{i=1}^{N} u_i(t,t_0,u_0) < b(\varepsilon),\ t \ge t_0 \tag{5.1}$$

where $u(t,t_0,u_0)$ is any solution of (3.5).

Choose $u_0 = \Omega(t_0^+, \gamma_0)$ and

$$\sum_{i=1}^{N} u_{i0} = a(t_0, \|\gamma_0\|)$$

Since $a(t, \mathrm{K})$ and $a \in C[R_+ \times R_+, R_+]$, we can find a positive function $\delta = \delta(t_o, \delta_1) > 0$ such that the inequalities

$$a(t_0, \|\gamma_0\|) < \delta_1\ \ and\ \ \|\gamma_0\| < \delta \tag{5.2}$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta$ then $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon$.

Suppose that this claim is false, then there would exists a point $t_1 \in [t_0,t)$ and the solution $\gamma(t,t_0,\gamma_0)$ with $\|\gamma_0\| < \delta_1$ such that

$$\|\gamma(t_1)\| = \varepsilon \ \ and \ \ \|\gamma(t)\| < \varepsilon \ \text{for} \ \ t \in [t_0,t_1) \tag{5.3}$$

This implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ for $t \in [t_0,t_1)$

So that using equations (4.1) and condition (iv) of Theorem 5.1 we have the estimate

$$b(\|\gamma(t_1)\|) \le \sum_{i=1}^{N} \Omega_i(t_1,\gamma(t_1)) \text{ , implying}$$

$$b(\varepsilon) \le \sum_{i=1}^{N} \Omega_i(t_1,\gamma(t_1)) \le \vartheta(t) \tag{5.4}$$

where $\vartheta(t) = \sum_{i=1}^{n} \vartheta_i(t,t_0,u_0)$ is the maximal solution of (3.5).

Then, using equations (5.4), (5.3) and condition (iii) of Theorem 5.1 we arrive at the estimate

$$b(\varepsilon) \le \Omega_0(t_1,\gamma(t_1)) \le \sum_{i=1}^{N} \vartheta_i(t,t_0,u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

## 6 Application

Consider the system of fractional differential equations

$$\begin{aligned}
{}^{C}D^{\beta}\gamma_1(t) &= -15\gamma_1 - \tfrac{\gamma_2^2 \cos ec\gamma_1}{2\gamma_1} + \gamma_1 \sin \gamma_2 + \tfrac{3\gamma_2^2 \sin\gamma_1}{\gamma_1}, \ t \ne t_k \\
{}^{C}D^{\beta}\gamma_2(t) &= \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin \gamma_1 - 5\gamma_2 \cos ec\gamma_1 - \gamma_1^2 \cos\gamma_2, \ \ t \ne t_k \\
\Delta\gamma_1 &= \xi_k(\gamma(t_k)), \ t = t_k \\
\Delta\gamma_2 &= \tau_k(\gamma(t_k)), \ \ t = t_k \ , \ k = 1,2,...
\end{aligned} \tag{6.1}$$

with initial conditions

$$\gamma_1(t_0^+) = \gamma_{10} \ \ and \ \gamma_2(t_0^+) = \gamma_{20}$$

where $\gamma_1, \gamma_2 \in R^N$ are arbitrary functions.

Equation (6.1) is equivalent to (3.3) and $f = (f_1 f_2)$, where

$$f_1(t,\gamma_1) = {}^{C}D^{\beta}\gamma_1(t) = -15\gamma_1 - \tfrac{\gamma_2^2 \cos ec\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{3\gamma_2^2 \sin\gamma_1}{\gamma_1}$$ and

$$f_2(t,\gamma_2) = {}^{C}D^{\beta}\gamma_2(t) = \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin\gamma_1 - 5\gamma_2 \cos ec\gamma_1 - \gamma_1^2 \cos\gamma_2$$

Consider a vector Lyapunov function of the form $\Omega = (\Omega_1, \Omega_2)^T$, where

$$\Omega_1(t,\gamma_1,\gamma_2) = \gamma_1^{\ 2},\ \Omega_2(t,\gamma_1,\gamma_2) = \gamma_2^{\ 2}$$

So that $\Omega = (\Omega_1, \Omega_2)^T$ with $\gamma = (\gamma_1, \gamma_2) \in R^2$, so that $\|\gamma\| = \sqrt{\gamma^2 + y^2}$

$$\sum_{i=1}^{2} \Omega_i(t,\gamma_1,\gamma_2) = \gamma_1^2 + \gamma_2^2$$

The assumption,

$$b(\|\gamma\|) \le \sum_{i=1}^{n} \Omega_i(\gamma, y) \le a(t, \|\gamma\|) \text{ reduces to}$$

$$\sqrt{\gamma^2 + y^2} \le \gamma^2 + y^2 \le 2\left(\sqrt{\gamma^2 + y^2}\right)^2$$

with the proviso that $b(\wp) = \wp, and\ a(\wp) = 2\wp^2$.

Furthermore, we deduce that using equation (3.4) and $\Omega_1(t,\gamma_1,\gamma_2) = \gamma_1^{\ 2}$

$${}^{C}D_+^{\beta}\Omega(t,\gamma_1,\gamma_2) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\Omega(t,\gamma_1,\gamma_2) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\, {}^{\beta}C_r \left[\Omega(t-rh, \gamma - h^\beta f_i(t,\gamma_1,\gamma_2)) - \Omega(t_0,\gamma_0)\right]\right\}, t \ge t_0$$

$${}^{C}D_+^{\beta}\Omega_1(t,\gamma_1) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1^{\ 2} - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r} ({}^{\beta}C_r) \left[\Omega(t-rh, \gamma - h^\beta f_1(t,\gamma_1)) - \gamma_{10}^2\right]\right\}, t \ge t_0$$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r) \quad [(\gamma_1 - h^\beta f_1(t,\gamma_1))^2 - \gamma_{10}^2]\right\}, t \ge t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)[\gamma_1^2 - 2\gamma_1 h^\beta f_1(t,\gamma_1) + h^{2\beta} f_1{}^2(t,\gamma_1)) - \gamma_{10}^2]\right\}, t \ge t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)\gamma_1^2 - \gamma_{10}^2 - + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)\gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)h^\beta f_1(t,\gamma_1)\right\}, t \ge t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)\gamma_1^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)\gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r)h^\beta f_1(t,\gamma_1)\right\}, t \ge t_0 \qquad (6.2) $$

Recall from equations (3.7) and (3.8) in [1] that

$$ \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r) = \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} \ \textit{and} \ \lim_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^\beta C_r) = -1 \qquad (6.3) $$

and substituting (6.3) into (6.2), we have,

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} - \frac{\gamma_{10}^2}{t^\beta\Gamma(1-\beta)} + 2\gamma_1 f_1(t,\gamma_1) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} + 2\gamma_1 f_1(t,\gamma_1) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} + 2\gamma_1(-15\gamma_1 - \tfrac{\gamma_2^2 \operatorname{cos}ec\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{3\gamma_2^2\sin\gamma_1}{\gamma_1}) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} - 30\gamma_1^2 - \gamma_2^2 \cos ec\gamma_1 + 2\gamma_1^2 \sin\gamma_2 + 6\gamma_2^2 \sin\gamma_1 $$

$As\ t \to \infty,\ \dfrac{\gamma_1^2}{t^\beta\Gamma(1-\beta)} \to 0,$ so we now have that

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le -30\gamma_1^2 - \gamma_2^2 \cos ec\gamma_1 + 2\gamma_1^2 \sin\gamma_2 + 6\gamma_2^2 \sin\gamma_1 $$

$$ = 2\gamma_1^2(-15 + \sin\gamma_2) + \gamma_2^2(6\sin\gamma_1 - \cos ec\gamma_1) $$

$$\leq 2\gamma_1^2(-15+|\sin\gamma_2|)+\gamma_2^2(6|\sin\gamma_1|-\tfrac{1}{|\sin\gamma_1|})$$

$$\leq 2\gamma_1^2(-15+1)+\gamma_2^2(6-1)$$

$$=\gamma_1^2(-14)+\gamma_2^2(5)$$

Therefore,

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\leq -14\Omega_1+5\Omega_2 \tag{6.4}$$

Also for $\gamma_0 \in S_\psi$, $for\ t=t_k,\ k=1,2,...$ we have, $\Omega_1(t,\gamma(t)+\xi_k)=\left|\xi_k+\gamma(t)\right|\leq\Omega_1(t,\gamma(t))$

Similarly, we compute for the Dini derivative for $\Omega_2(t,\gamma_2)=\gamma_2^2$ and follow through the same argument by substituting for $f_2(t,\gamma_2)={}^{C}D^{\beta}\gamma_2(t)=\frac{3\gamma_1^2}{\gamma_2}-\gamma_2\sin\gamma_1-5\gamma_2\cos ec\gamma_1-\gamma_1^2\cos\gamma_2$ in equation (3.4) to have that,

$$^{C}D_{+}^{\beta}\Omega_2(t,\gamma_2)\leq 4\Omega_1-12\Omega_2 \tag{6.5}$$

Also for $\gamma_0 \in S_\psi$, $for\ t=t_k,\ k=1,2,...$ we have, $\Omega_2(t,\gamma(t)+\tau_k)=\left|\tau_k+\gamma(t)\right|\leq\Omega_2(t,\gamma(t))$

By combining equations (6.4) and (6.5), we have

$$^{C}D_{+}^{\beta}\Omega\leq\begin{pmatrix}-14 & 5\\ 4 & -12\end{pmatrix}\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix}=g(t,\Omega) \tag{6.6}$$

Now, consider the comparison system $^{C}D^{\beta}u=g(t,\Omega)=Au$, where $A=\begin{pmatrix}-14 & 5\\ 4 & -12\end{pmatrix}$.

The vectorial inequality (6.6) and all other conditions of Theorem 5.2 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (6.1) is uniformly practically stable. Therefore, the trivial solution $\gamma_0=0$ of the IFrDE (6.1) is uniformly practically stable.

## 7 Conclusion

In this paper, we examined the uniform practical stability of nonlinear impulsive Caputo fractional differential equations with fixed moment of impulse, using an auxiliary Lyapunov functions which are analogues of vector Lyapunov functions, and is generalized by a class of piecewise continuous Lyapunov functions. By splitting the vector Lyapunov functions $V(t,\gamma)$ into several components of $V_1, V_2,...,$ each of the state variable or solution state $x_1, x_2,..., x_m$ can be inserted into each of the components rather than into one Lyapunov function, so that the arguments to obtained the desired practical stability result becomes less complicated, and the imposed conditions would be less restrictive. Together with the comparison results, sufficient conditions for the uniform practical stability of the impulsive Caputo fractional order system (3.3) is established with an illustrative example.

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