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On the Auxiliary Lyapunov Functions and Uniform Practical Stability for Nonlinear Impulsive Caputo Fractional Order Derivative via New Modelled Generalized Dini Derivative

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Abstract.

In this paper, the uniform practical stability of the trivial solution of a nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse is examined using an auxiliary Lyapunov function which are analogues of vector Lyapunov functions, and is generalized by a class of piecewise continuous Lyapunov functions. Together with comparison results, sufficient conditions for the uniform practical stability for impulsive Caputo fractional order systems are established. An illustrative example is given to confirm the suitability of the obtained results.

Keywords: Uniform practical stability, Caputo derivative, impulse, Lyapunov function.

MSC: 34A08; 34A34; 34A37; 34D20.

1 INTRODUCTION

Spanning three centuries, the history of fractional calculus has been marked by significant developments, largely confined to the theoretical foundations of mathematics. Moreover, recently, its relevance have also span into several other fields like the mathematical physics,

engineering, control theory, economics, etc. The earliest – more or less systematic studies of the concept seem to have been made in the 19th century by Liouville, Riemann, Caputo, etc. [37], and in the last four decades, it was observed that the advancement of fractional calculus has enabled the description of complex system behaviors through fractional differential systems (including many physical phenomenon having memory and genetic characteristics), thus providing new insights into their dynamics.

As observed in [6, 27], one of the trends in the stability theory of solutions of differential equations is the so-called practical stability. This aspect of stability was introduced by LaSalle and Lefschetz in [24], and it is used in estimating the worst-case transient and steady-state responses together with verifying pointwise in time constraints imposed on the trajectory curve. Fundamental results in this area were established in [9, 31] and [40] for integer order derivative.

In line with the development of the theory of practical stability in recent years is the mathematical theory of impulsive differential equations which have experienced a massive research attention and development. Now, the theory of impulsive differential equations is richer than the corresponding theory of differential equations [16] as they constitute very important models for describing the true state of several real life processes and phenomena, since many evolution processes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly (see [16]).

Moreover, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [35], and one of the most versatile methods in the study of the stability properties of impulsive systems is the Lyapunov's direct method also called the Lyapunov's second method. Interestingly, the method involves seeking an appropriate continuously differentiable function (called Lyapunov's function) that is positive definite and whose time derivative along the solution path or trajectory curve is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [12] and [30]. Investigations into the stability of fractional order systems is a recent development and major obstacles in utilizing the Lyapunov's functions for these systems lies in establishing an appropriate definition of the derivative amidst fractional differential equations. However, this difficulty was safely handled in [1] where the choice of a Lyapunov function that is continuous was adopted, with a re-definition, as it were, of the Dini derivative for the given fractional order system with respect to the Lyapunov function. To the best of our knowledge, not so much have been done in the literature in terms of practical stability for impulsive fractional order derivative using the vector Lyapunov functions by means of the comparison method.

Against this background, [6] established sufficient conditions for the practical stability of impulsive fractional order systems using the vector Lyapunov functions with the comparison method, while fundamental results for the stability of impulsive Caputo fractional differential

equations using the scalar Lyapunov function were examined in [1] and [2]. Other stability results have been established in [42-48].

In this paper, the uniform practical stability of impulsive Caputo fractional order systems is examined. By means of the comparison principle, sufficient conditions for the uniform practical stability of impulsive fractional order systems is established using a class of piecewise continuous functions. An illustrative example given, extends and improves on existing results.

2 Preliminary Notes and Basic Definitions

Fractional calculus is considered as a natural generalization of the classical calculus of integer order to arbitrary order, thereby allowing for the extension of the traditional concepts of derivative and integral to functions with fractional orders. By this extension, functions with non integer orders are much more flexible in describing real world systems (see [14, 25, 26, 34 and 37]).

There are several definitions of fractional derivatives and fractional integrals.

General case. Let the number $n-1 < \beta < n, \beta > 0$ be given, where n is a natural number, and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.1.

The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$${}^{RL}D_t^\beta \gamma(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{n-\beta-1} \gamma(s) ds, t \geq t_0$$

Definition 2.2.

The Caputo fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$${}^CD_t^\beta \gamma(t) = \frac{1}{\Gamma(n-\beta)} \int_{t_0}^t (t-s)^{n-\beta-1} \gamma^{(n)}(s) ds, t \geq t_0$$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3.

The Grunwald-Letnikov fractional derivative of order β of $\gamma(t)$ is given by (See [1])

$${}^{GL}D_0^\beta \gamma(t) = \lim_{h \rightarrow 0^+} \frac{1}{h^\beta} \sum_{r=0}^{\left\lfloor \frac{t-t_0}{h} \right\rfloor} (-1)^r {}^\beta C_r \gamma(t-rh), \quad t \geq t_0$$

and

Definition 2.4. The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (See [1])

$${}^{GL}D_0^\beta \gamma(t) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h^\beta} \sum_{r=0}^{\left\lfloor \frac{t-t_0}{h} \right\rfloor} (-1)^r {}^\beta C_r \gamma(t-rh), \quad t \geq t_0$$

where ${}^\beta C_r$ are the binomial coefficients and $\left\lfloor \frac{t-t_0}{h} \right\rfloor$ denotes the integer part of $\frac{t-t_0}{h}$.

Particular case (when $n=1$). In most applications, the order of β is often less than 1, so that $\beta \in (0,1)$. For simplicity of notation, we will use ${}^C D^\beta$ instead of ${}_{t_0}^C D^\beta$ and the Caputo fractional derivative of order β of the function $\gamma(t)$ is

$${}^C D^\beta \gamma(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{-\beta} \gamma'(s) ds, \quad t \geq t_0$$

(2.1)

3 Impulses in Fractional Dynamics

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < \beta < 1$.

$$\begin{aligned} {}^C D^\beta \gamma(t) &= f(t, \gamma), \quad t \geq t_0, \\ \gamma(t_0) &= \gamma_0, \end{aligned}$$

(3.1)

where $\gamma \in R^N$, $f \in C[R_+ \times R^N, R^N]$, $f(t, 0) \equiv 0$ and $(t_0, x_0) \in R_+ \times R^N$.

Some sufficient conditions for the existence of the global solutions to (3.1) are considered in [7], [11], [22], [23], [29], [35], [42].

The IVP for FrDE (3.1) is equivalent to the following Volterra integral equation (See [2]),

$$\gamma(t) = \gamma_0 + \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} f(s, \gamma(s)) ds, \quad t \geq t_0$$

(3.2)

Consider the IVP for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < \beta < 1$,

$$\begin{aligned} {}^c D^\beta \gamma(t) &= f(t, \gamma), \quad t \geq t_0, \quad t \neq t_k, \quad k = 1, 2, \dots \\ \Delta \gamma &= I_k(\gamma(t_k)), \quad t = t_k, \quad k \in N, \\ \gamma(t_0^+) &= \gamma_0, \end{aligned} \quad (3.3)$$

where $\gamma, \gamma_0 \in R^N$, $f \in C[R_+ \times R^N, R^N]$, and $t_0 \in R_+$, $I_k : R^N \rightarrow R^N$, $k = 1, 2, \dots$

under the following assumptions:

- (i) $0 < t_1 < t_2 < \dots < t_k < \dots$, and $t_k \rightarrow \infty$ as $k \rightarrow \infty$;
- (ii) $f : R_+ \times R^N \rightarrow R^N$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $x \in R^N$, $k = 1, 2, \dots$, and $\lim_{(t, \gamma) \rightarrow (t_k^+, \gamma)} f(t, \gamma) = f(t_k^+, \gamma)$ exists;
- (iii) $I_k : R^N \rightarrow R^N$

In this paper, we assume that $f(t, 0) \equiv 0$, $I_k(0) \equiv 0$ for all k so that we have trivial solution for (3.3), and the points t_k , $k = 1, 2, \dots$ are fixed such that $t_1 < t_2 < \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$. The system (3.3) with initial condition $\gamma(t_0) = \gamma_0$ is assumed to have a solution $\gamma(t; t_0, \gamma_0) \in PC^\beta([t_0, \infty), R^N)$.

Remark 3.1. The second equation in (3.3) is called the impulsive condition, and the function $I_k(\gamma(t_k))$ gives the amount of jump of the solution at the point t_k .

Definition 3.1. Let $\Omega : R_+ \times R^N \rightarrow R^N$. Then Ω is said to belong to class χ if,

- (i) Ω is continuous in $(t_{k-1}, t_k]$ and for each $\gamma \in R^N$ and $\lim_{(t, \gamma) \rightarrow (t_k^+, \gamma)} \Omega(t, \gamma) = \Omega(t_k^+, \gamma)$ exists;
- (ii) Ω is locally Lipschitz with respect to its second argument x and $\Omega(t, 0) \equiv 0$

Now, for any function $\Omega(t, \gamma) \in PC([t_0, \infty) \times \xi, R_+^N)$. we define the Caputo fractional Dini derivative as:

$${}^c D_+^\beta \Omega(t, \gamma) = \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \Omega(t, \gamma) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h} \right]} (-1)^{r+1} \beta C_r [\Omega(t-rh, \gamma - h^\beta f(t, \gamma) - \Omega(t_0, \gamma_0))] \right\} \quad (3.4)$$

$t \geq t_0$ where $t \in [t_0, \infty)$, $\gamma, \gamma_0 \in \xi$, $\xi \in R^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T]$.

Definition 3.2. A function $g \in PC[R^n, R^n]$ is said to be quasimonotone nondecreasing in γ , if $\gamma \leq y$ and $\gamma_i = y_i$ for $1 \leq i \leq n$ implies $g_i(\gamma) = g_i(y)$.

Definition 3.3. The zero solution of (3.3) is said to be:

(PS1) practically stable if for every $\varepsilon > 0$ and $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \leq \delta$ implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \geq t_0$;

(PS2) uniformly practically stable if for every $\varepsilon > 0$ and $t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$, continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \leq \delta$ implies $\gamma_0 \in R^N$ $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$ for $t \geq t_0$;

(PS3) asymptotically practically stable if it is practically stable and if for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \geq t_0 + T$ and $\|\gamma_0\| \leq \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$;

(PS4) uniformly asymptotically practically stable if it is uniformly practically stable and $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that for $t \geq t_0 + T$, the inequality $\|\gamma_0\| \leq \delta$ implies $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Definition 3.4. A function $a(r)$ is said to belong to the class K if $a \in PC([0, \psi), R_+)$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing in r .

In this paper, we define the following sets:

$$\bar{S}_\psi = \{\gamma \in R^N : \|\gamma\| \leq \psi\}$$

$$S_\psi = \{\gamma \in R^N : \|\gamma\| < \psi\}$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned}
{}^C D^\beta u &= g(t, u), \quad t \geq t_0, \quad t \neq t_k, \quad k = 1, 2, \dots \\
\Delta u &= \psi_k(u(t_k)), \quad t = t_k, \quad k \in N, \\
u(t_0^+) &= u_0,
\end{aligned}
\tag{3.5}$$

existing for $t \geq t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g : R_+ \times R_+^n \rightarrow R^n$, $g(t, 0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n . The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (3.5) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC^\beta([t_0, \infty), R^n)$.

Lemma 3.2. Assume $m \in PC([t_0, T] \times \bar{S}_\psi, R^N)$. and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^* , then the inequality ${}^C D_+^\beta m(t^*, \alpha_1) - {}^C D_+^\beta m(t^*, \alpha_2) > 0$ holds.

Proof. Let $\Omega(t, \gamma) = m(t, \alpha_1) - m(t, \alpha_2)$. Applying (3.4), we have

$$\begin{aligned}
{}^C D_+^\beta (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \{ [m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{\left[\frac{t-t_0}{h} \right]} (-1)^{r+1} C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \}
\end{aligned}$$

when $\alpha_1 = \alpha_2$ we have

$$\begin{aligned}
{}^C D_+^\beta (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \{ -[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{\left[\frac{t-t_0}{h} \right]} (-1)^{r+1} C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \}
\end{aligned}$$

$$\begin{aligned}
{}^C D_+^\beta (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= - \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad + \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \sum_{r=1}^{\left[\frac{t-t_0}{h} \right]} (-1)^r C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&\quad - \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \sum_{r=1}^{\left[\frac{t-t_0}{h} \right]} (-1)^r C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)]
\end{aligned}$$

$$\begin{aligned}
{}^C D_+^\beta (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= -\lim_{h \rightarrow 0^+} \sup \frac{1}{h^\beta} [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \lim_{h \rightarrow 0^+} \sup \frac{1}{h^\beta} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^r C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
{}^C D_+^\beta (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= -\lim_{h \rightarrow 0^+} \sup \frac{1}{h^\beta} \sum_{r=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^r C_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)]
\end{aligned}$$

Applying equation (3.8) in [1], we have

$${}^C D_+^\beta m(t^*, \alpha_1) - {}^C D_+^\beta m(t^*, \alpha_2) = -\frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)} [m(t^*, \alpha_1) - m(t^*, \alpha_2)]$$

By the lemma, we have that

$$m(t, \alpha_1) - m(t, \alpha_2) < 0 \quad \text{for } t_0 \leq t \leq t^*$$

And so, it follows that

$${}^C D_+^\beta m(t^*, \alpha_1) - {}^C D_+^\beta m(t^*, \alpha_2) > 0$$

Note that some existence results for (3.5) are given in [11, 13] and [14].

Remark 3.3. Lemma 3.2 extends Lemma 1 in [1], where the vectors $m(t, \alpha_1)$ and $m(t, \alpha_2)$ are compared component-wise.

In the following, we establish the comparison result for the system (3.3).

4 Fractional Differential Inequalities and Comparison Results for Vector Fractional Differential Equation

In this section, again we assume that $0 < \beta < 1$.

Theorem 4.1. (Comparison Result)

Assume that

- i) $g \in PC[R_+ \times R_+^n, R^n]$ and is continuous in $(t_{k-1}, t_k]$, $k=1,2,\dots$ and $g(t, u)$ is quasimonotone non-decreasing in u for each $u \in R^n$ and $\lim_{(t,y) \rightarrow (t_k^+, u)} g(t, u) = g(t_k^+, u)$ exists;
- ii) $\Omega \in PC[R_+ \times R^N, R_+^N]$ and $\Omega \in \mathcal{X}$ such that

$${}^C D_+^\beta \Omega(t, \gamma) \leq g(t, \Omega(t, \gamma)), \quad (t, \gamma) \in R_+ \times R^N$$

and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \leq \omega_k(\Omega(t, \gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \rightarrow R_+^N$ is nondecreasing for $k = 1, 2, \dots$

- iii) $\mathcal{G}(t) = \mathcal{G}(t, t_0, u_0) \in PC^\beta([t_0, T], R^n)$ be the maximal solution of the IVP for the IFrDE system (3.5)

Then,

$$\Omega(t, \gamma(t)) \leq \mathcal{G}(t), \quad t \geq t_0, \quad (4.1)$$

where $\gamma(t) = \gamma(t, t_0, \gamma_0) \in PC^\beta([t_0, T], R^N)$ is any solution of (3.3) existing on $[t_0, \infty)$, provided that

$$\Omega(t_0^+, \gamma_0) \leq u_0 \quad (4.2)$$

Proof.

Let $\eta \in S_\psi$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations,

$$\begin{aligned} {}^C D^\beta u &= g(t, u) + \eta, \quad \text{for } t \in [t_0, \infty) \\ u(t_0^+) &= u_0 + \eta, \end{aligned} \quad (4.3)$$

for $t \in [t_0, \infty)$.

The function $u_\eta(t, \alpha)$ is a solution of (4.3), where $\alpha > 0$ the fractional differential equation (3.5) if and only if it satisfies the Volterra fractional integral equation,

$$u_\eta(t, \alpha) = u_0 + \eta + \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} (g(s, u_\eta(s, \alpha)) + \alpha) ds, \quad t \in [t_0, \infty). \quad (4.4)$$

Let the function $m(t, \alpha) \in PC([t_0, T] \times S_\psi, R_+^N)$ be defined as $m(t, \alpha) = \Omega(t, \gamma^*(t))$

We now prove that

$$m(t, \alpha) < u_\eta(t, \alpha) \text{ for } t \in [t_0, \infty) \quad (4.5)$$

Observe that the inequality (4.5) holds whenever $t = t_0$ i.e.

$$m(t_0, \alpha) = \Omega(t_0, \gamma_0) \leq u_0 < u_\eta(t_0, \alpha)$$

Assume that the inequality (4.5) is not true, then there exists a point $t_1 > t_0$ such that

$$m(t_1, \alpha) = u_\eta(t_1, \alpha) \text{ and } m(t, \alpha) < u_\eta(t, \alpha) \text{ for } t \in [t_0, t_1).$$

It follows from Lemma 3.2 that

$${}^C D_+^\beta (m(t_1, \alpha) - u_\eta(t_1, \alpha)) > 0 \text{ i.e.}$$

$${}^C D_+^\beta (m(t_1, \alpha)) > {}^C D_+^\beta u_\eta(t_1, \alpha)$$

$${}^C D_+^\beta \Omega(t_1, \gamma(t_1)) > {}^C D_+^\beta u_\eta(t_1, \alpha)$$

and using (4.3) we arrive at

$${}^C D_+^\beta \Omega(t_1, \gamma(t_1)) > g(t_1, u(t_1, \alpha)) + \eta > g(t_1, u(t_1, \alpha))$$

Therefore,

$${}^C D_+^\beta m(t_1, \alpha) > g(t_1, u(t_1, \alpha)) \quad (4.6)$$

From Theorem 4.1, the function $\gamma^*(t) = \gamma(t, t_0, \gamma_0)$ satisfies the IVP (4.3) and the equality

$$\limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} [\gamma^*(t) - \gamma_0 - S(\gamma^*(t), h)] = f(t, \gamma^*(t)), \text{ holds} \quad (4.7)$$

where $\gamma^*(t) = \gamma(t, t_0, \gamma_0)$ is any other solution of (3.5), and

$$S(\gamma^*(t), h) = \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r [\gamma^*(t - rh) - \gamma_0] \quad (4.8)$$

is the Grunwald Letnikov fractional derivative

Multiply equation (4.7) through by h^β

$$\begin{aligned}\limsup_{h \rightarrow 0^+} [\gamma^*(t) - \gamma_0 - S(\gamma^*(t), h)] &= h^\beta f(t, \gamma^*(t)) \\ \gamma^*(t) - \gamma_0 - [S(\gamma^*(t), h) + \rho(h^\beta)] &= h^\beta f(t, \gamma^*(t)) \\ \gamma^*(t) - h^\beta f(t, \gamma^*(t)) &= [S(\gamma^*(t), h) + \gamma_0 + \rho(h^\beta)]\end{aligned}\tag{4.9}$$

Then for $t \in [t_0, \infty)$, we have

$$\begin{aligned}m(t, \alpha) - m(t_0, \alpha) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} C_r [m(t-rh, \alpha) - m(t_0, \alpha)] &= \\ \Omega(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} C_r [\Omega(t-rh, \gamma^*(t) - h^\beta f(t, \gamma^*(t)) - \Omega(t_0, \gamma_0)] & \\ = \Omega(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} C_r [\Omega(t-rh) - \gamma^*(t) - h^\beta f(t, \gamma^*(t)) - \Omega(t_0, \gamma_0)] &+ \\ \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} C_r \{[\Omega(t-rh), S(\gamma^*(t), h) + \gamma_0 + \rho(h^\beta) - \Omega(t_0, \gamma_0)] - [\Omega(t-rh, \gamma^*(t-rh)) - \Omega(t_0, \gamma_0)]\} & \\ (4.10) &\end{aligned}$$

Since $\Omega(t, \gamma)$ is locally Lipschitzian with respect to the second variable, we have that,

$$\leq L \left| (-1)^{r+1} \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (\beta C_r) [S(\gamma^*(t), h) + \gamma_0 + \rho(h^\beta) - \gamma^*(t-rh)] \right\| \right|$$

where $L > 0$ is a Lipschitz constant.

$$\begin{aligned}&\leq L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (\beta C_r) [S(\gamma^*(t), h) + \rho(h^\beta)] - (\gamma^*(t-rh) - \gamma_0) \right\| \\ (4.11) &\end{aligned}$$

Using equation (4.8), equation (4.11) becomes,

$$\leq L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (\beta C_r) \left(\sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} (\beta C_r) [(\gamma^*(t-rh) - \gamma_0) + \rho(h^\beta)] - (\gamma^*(t-rh) - \gamma_0) \right) \right\|$$

$$\begin{aligned}
&\leq L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} ({}^\beta C_r) (-1)^{r+1} \left[\sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} {}^\beta C_r [(\gamma^*(t-rh) - \gamma_0)] + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} {}^\beta C_r \rho(h^\beta) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} {}^\beta C_r (\gamma^*(t-rh) - \gamma_0) \right] \right\| \\
&\leq L (-1)^{r+1} \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} ({}^\beta C_r) [(\gamma^*(t-rh) - \gamma_0)] + \left[\sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} {}^\beta C_r - 1 \right] + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} {}^\beta C_r \rho(h^\beta) \right\|
\end{aligned}$$

(4.12)

Substituting equation (4.12) into (4.10) we have

$$\begin{aligned}
&= \Omega(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \left[\Omega(t-rh) - \gamma^*(t) - h^\beta f(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) \right] + \\
&L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} ({}^\beta C_r) (\gamma^*(t-rh) - \gamma_0) \left[\sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r - 1 \right] + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \rho(h^\beta) \right\| \\
&= \Omega(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \left[\Omega(t-rh) - \gamma^*(t) - h^\beta f(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) \right] + \\
&L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} ({}^\beta C_r) (\gamma^*(t-rh) - \gamma_0) \left[- \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r {}^\beta C_r - 1 \right] + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \rho(h^\beta) \right\|
\end{aligned}$$

Dividing through by $h^\beta > 0$ and taking the $\limsup$ as $h \rightarrow 0^+$ we have,

$$\begin{aligned}
{}^C D_+^\beta m(t, \alpha) &= \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \Omega(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \left[\Omega(t-rh) - \gamma^*(t) - h^\beta f(t, \gamma^*(t)) - \Omega(t_0, \gamma_0) \right] \right\} + \\
&\limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} ({}^\beta C_r) (\gamma^*(t-rh) - \gamma_0) \left[- \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r {}^\beta C_r - 1 \right] + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} {}^\beta C_r \rho(h^\beta) \right\|
\end{aligned}$$

Recall that,

$$\lim_{h \rightarrow 0^+} \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} ({}^\beta C_r) = -1 \text{ and } \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \rho(h^\beta) = 0$$

From equations (3.6) and (3.7) in [1], we have that

$${}^C D_+^\beta m(t, \alpha) = {}^C D_+^\beta \Omega(t, \gamma^*(t)) + L \left\| \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} ({}^\beta C_r) (\gamma^*(t-rh) - \gamma_0) [-(-1) - 1] + 0 \right\|$$

Using condition (ii) of the Theorem 4.1, we obtain the estimate

$${}^c D_+^\beta m(t, \alpha) \leq g(t, \Omega(t, \gamma^*(t))) = g(t, m(t, \alpha)), \quad (4.13)$$

Also,

$$m(t_0^+, \alpha) \leq u_0 \text{ and } m(t_k^+, \alpha) = \Omega(t_k^+, \gamma(t_k)) + I_k(\gamma(t_k)) \leq \rho_k(m(t_k)) \quad (4.14)$$

Now equation (4.14) with $t = t_1$ contradicts (4.6), hence (4.5) is true.

□

For $t \in [t_0, T]$, we now establish that

$$u_{\eta_1}(t, \alpha) < u_{\eta_2}(t, \alpha) \text{ whenever } \eta_1 < \eta_2 \quad (4.15)$$

Observe that the inequality (4.15) holds for $t = t_0$

Assume that (4.15) is not true. Then there exists a point t_1 such that $u_{\eta_1}(t_1, \alpha) = u_{\eta_2}(t_2, \alpha)$ and $u_{\eta_1}(t, \alpha) < u_{\eta_2}(t, \alpha)$ for $t \in [t_0, t_1)$.

By Lemma 3.2, we have that

$${}^c D_+^\beta (u_{\eta_1}(t_1, \alpha) - u_{\eta_2}(t_2, \alpha)) > 0$$

However,

$$\begin{aligned} {}^c D_+^\beta (u_{\eta_1}(t_1, \alpha) - u_{\eta_2}(t_2, \alpha)) &= {}^c D_+^\beta (u_{\eta_1}(t_1, \alpha) - u_{\eta_2}(t_2, \alpha)) \\ &= g(t_1, u(t_1, \alpha)) + \eta_1 - [g(t_1, u(t_1, \alpha)) + \eta_2] \\ &= \eta_1 - \eta_2 < 0 \end{aligned}$$

which is a contradiction, and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_\eta(t, \alpha)\}$, $t \in [t_0, T]$ of (4.3) is uniformly bounded, i.e. there exists $\lambda > 0$ with $|u_\eta(t, \alpha)| \leq \lambda$, with bound λ on $[t_0, T]$. We now show that the family $\{u_\eta(t, \alpha)\}$ is equicontinuous on $[t_0, T]$. Let $k = \sup \{ |g(t, \gamma)| : (t, \gamma) \in [t_0, T] \times [-\lambda, \lambda] \}$, where λ is the bound on the family $\{u_\eta(t, \alpha)\}$. Fix a decreasing sequence $\{\eta_i\}_{i=0}^\infty(t)$, such that $\lim_{i \rightarrow \infty} \eta_i = 0$ and consider a

sequence of functions $\{u_\eta(t, \alpha)\}$. Again, let $t_1, t_2 \in [t_0, T]$, with $t_1 < t_2$, then we have the following estimates,

$$\begin{aligned}
\|u_\eta(t_2, \alpha) - u_\eta(t_1, \alpha)\| &\leq \|u_0 + \eta_i + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_2} (t_2 - s)^{\beta-1} (g(s, u(s, \eta_i)) + \eta_i) ds \\
&\quad - \left(u_0 + \eta_i + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} (g(s, u(s, \eta_i)) + \eta_i) ds \right)\| \\
&\leq \frac{1}{\Gamma(\beta)} \left\| \left[\int_{t_0}^{t_2} (t_2 - s)^{\beta-1} - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} \right] (g(s, u(s, \eta_i)) + \eta_i) ds \right\| \\
&\leq \frac{k}{\Gamma(\beta)} \left\| \left(\int_{t_0}^{t_2} (t_2 - s)^{\beta-1} - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} \right) ds \right\| \\
&\leq \frac{k}{\Gamma(\beta)} \left\| \left(\int_{t_0}^{t_1} (t_2 - s)^{\beta-1} + \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} \right) ds \right\| \\
&\leq \frac{k}{\Gamma(\beta)} \left\| \int_{t_0}^{t_1} (t_2 - s)^{\beta-1} - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} + \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} ds \right\| \\
&\leq \frac{k}{\Gamma(\beta)} \left[\left\| \int_{t_0}^{t_1} (t_2 - s)^{\beta-1} - \int_{t_0}^{t_1} (t_1 - s)^{\beta-1} ds \right\| + \left\| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} ds \right\| \right] \\
&\leq \frac{k}{\beta\Gamma(\beta)} \left[\|(t_2 - t_0)^\beta - (t_1 - t_0)^\beta - (t_2 - t_1)^\beta\| + \|(t_2 - t_1)^\beta\| \right] \\
&\leq \frac{k}{\beta\Gamma(\beta)} \left[(t_2 - t_1)^\beta + (t_2 - t_1)^\beta \right] = \frac{2k}{\Gamma(\beta + 1)} (t_2 - t_1)^\beta
\end{aligned}$$

provided $\|t_2 - t_1\| < \delta(\varepsilon) = \left(\frac{\Gamma(\beta + 1)\varepsilon}{2k} \right)^{\frac{1}{\beta}}$, proving that the family of solutions $\{u_\eta(t, \alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t, \alpha)\}$ guarantees the existence of a subsequence $\{u_{\eta_{ij}}(t, \alpha)\}$ which converges uniformly to the function $\mathcal{G}(t)$ on $[t_0, T]$. Then we show that $\mathcal{G}(t)$ is a solution of (4.4). Thus, equation (4.4) becomes

$$u_{\eta_{i_j}}(t, \alpha) = u_{0i_j} + \eta_{i_j} + \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} (g_{i_j}(s, u_{\eta_{i_j}}(s, \alpha)) + \eta_{i_j}) ds, \quad t \in [t_0, \infty).$$

(4.16)

Taking the $\lim$ as $i_j \rightarrow \infty$ in (4.16) yields,

$$\mathcal{G}(t) = u_0 + \eta_{i_j} + \frac{1}{\Gamma(\beta)} \int_{t_0}^t (t-s)^{\beta-1} (g(s, \mathcal{G}(t))) ds, \quad t \in [t_0, \infty).$$

(4.17)

Hence, $\mathcal{G}(t)$ is a solution of (3.5) on $[t_0, T]$. We claim that $\mathcal{G}(t)$ is the maximal solution of (3.5). Then, from (4.5), we have that $m(t, \alpha) < u_{\eta}(t, \alpha) \leq \mathcal{G}(t)$ on $[t_0, T]$.

Suppose that in Theorem 4.1, $g(t, u) \equiv 0$, then we have the following results

Corollary 4.1.

Assume that Condition (i) of Theorem 4.1 holds and,

(i) $\Omega \in PC[R_+ \times R^N, R_+^N]$ and $\Omega \in \chi$ such that

$${}^C D_+^\beta \Omega(t, \gamma) \leq 0$$

(4.14)

holds, and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \leq \omega_k(\Omega(t, \gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \rightarrow R_+^N$ is nondecreasing for $k = 1, 2, \dots$

Then for $t \in [t_0, \infty)$, the inequality

$$\Omega(t, \gamma(t)) \leq \Omega(t_0^+, \gamma_0) \text{ holds}$$

5 Main Results

In this section, we will obtain sufficient conditions for the uniform practical stability of the system (3.3). Again we assume $0 < \beta < 1$.

Theorem 5.1. Assume that

i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n$, $k = 1, 2, \dots$, and $\lim_{(t,y) \rightarrow (t_k^+, u)} g(t, y) = g(t_k^+, u)$ exists, and $g(t, u)$ is quasimonotone nondecreasing in u

ii) $\Omega \in PC[R_+ \times R^N, R_+^N]$ and $\Omega \in \chi$ such that

$${}^c D_+^\beta \Omega(t, \gamma) \leq g(t, \Omega(t, \gamma)), t \neq t_k, \text{ holds for all } (t, \gamma) \in R_+ \times S_\psi,$$

there exists a $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \leq \omega_k(\Omega(t, \gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \rightarrow R_+^N$ is nondecreasing for $k = 1, 2, \dots$

iii) $b(\|\gamma\|) \leq \Omega_0(t, \gamma) \leq a(\|\gamma\|)$, where $a, b \in K$ and $\Omega_0(t, \gamma) = \sum_{i=1}^N \Omega_i(t, \gamma)$.

Then the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $0 < \varepsilon < \psi$ and $t_0 \in R_+$ be given.

Assume that the solution (3.5) is uniformly practically stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^N u_{i0} < \delta \text{ implies } \sum_{i=1}^N u_i(t, t_0, u_0) < b(\varepsilon), t \geq t_0 \quad (5.1)$$

where $u(t, t_0, u_0)$ is any solution of (3.5).

Choose $u_0 = \Omega(t_0^+, \gamma_0)$ and

$$\sum_{i=1}^N u_{i0} = a(t_0, \|\gamma_0\|)$$

Since $a(t, K)$ and $a \in C[R_+ \times R_+, R_+]$, we can find a positive function $\delta = \delta(t_0, \delta_1) > 0$ such that the inequalities

$$a(t_0, \|\gamma_0\|) < \delta_1 \text{ and } \|\gamma_0\| < \delta \quad (5.2)$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta$ then $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Suppose that this claim is false, then there would exists a point $t_1 \in [t_0, t)$ and the solution $\gamma(t, t_0, \gamma_0)$ with $\|\gamma_0\| < \delta_1$ such that

$$\|\gamma(t_1)\| = \varepsilon \text{ and } \|\gamma(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \quad (5.3)$$

This implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ for $t \in [t_0, t_1)$

So that using equations (4.1) and condition (iv) of Theorem 5.1 we have the estimate

$$b(\|\gamma(t_1)\|) \leq \sum_{i=1}^N \Omega_i(t_1, \gamma(t_1)), \text{ implying} \\ b(\varepsilon) \leq \sum_{i=1}^N \Omega_i(t_1, \gamma(t_1)) \leq \mathcal{G}(t) \quad (5.4)$$

where $\mathcal{G}(t) = \sum_{i=1}^n \mathcal{G}_i(t, t_0, u_0)$ is the maximal solution of (3.5).

Then, using equations (5.4), (5.3) and condition (iii) of Theorem 5.1 we arrive at the estimate

$$b(\varepsilon) \leq \Omega_0(t_1, \gamma(t_1)) \leq \sum_{i=1}^N \mathcal{G}_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

6 Application

Consider the system of fractional differential equations

$$\begin{aligned} {}^C D^\beta \gamma_1(t) &= -15\gamma_1 - \frac{\gamma_2^2 \operatorname{cosec} \gamma_1}{2\gamma_1} + \gamma_1 \sin \gamma_2 + \frac{3\gamma_2^2 \sin \gamma_1}{\gamma_1}, \quad t \neq t_k \\ {}^C D^\beta \gamma_2(t) &= \frac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin \gamma_1 - 5\gamma_2 \operatorname{cosec} \gamma_1 - \gamma_1^2 \cos \gamma_2, \quad t \neq t_k \\ \Delta \gamma_1 &= \xi_k(\gamma(t_k)), \quad t = t_k \\ \Delta \gamma_2 &= \tau_k(\gamma(t_k)), \quad t = t_k, \quad k = 1, 2, \dots \end{aligned} \quad (6.1)$$

with initial conditions

$$\gamma_1(t_0^+) = \gamma_{10} \text{ and } \gamma_2(t_0^+) = \gamma_{20}$$

where $\gamma_1, \gamma_2 \in R^N$ are arbitrary functions.

Equation (6.1) is equivalent to (3.3) and $f = (f_1 f_2)$, where

$$f_1(t, \gamma_1) = {}^C D^\beta \gamma_1(t) = -15\gamma_1 - \frac{\gamma_2^2 \operatorname{cosec} \gamma_1}{2\gamma_1} + \gamma_1 \sin \gamma_2 + \frac{3\gamma_2^2 \sin \gamma_1}{\gamma_1} \quad \text{and}$$

$$f_2(t, \gamma_2) = {}^C D^\beta \gamma_2(t) = \frac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin \gamma_1 - 5\gamma_2 \operatorname{cosec} \gamma_1 - \gamma_1^2 \cos \gamma_2$$

Consider a vector Lyapunov function of the form $\Omega = (\Omega_1, \Omega_2)^T$, where

$$\Omega_1(t, \gamma_1, \gamma_2) = \gamma_1^2, \quad \Omega_2(t, \gamma_1, \gamma_2) = \gamma_2^2$$

So that $\Omega = (\Omega_1, \Omega_2)^T$ with $\gamma = (\gamma_1, \gamma_2) \in R^2$, so that $\|\gamma\| = \sqrt{\gamma^2 + y^2}$

$$\sum_{i=1}^2 \Omega_i(t, \gamma_1, \gamma_2) = \gamma_1^2 + \gamma_2^2$$

The assumption,

$$b(\|\gamma\|) \leq \sum_{i=1}^n \Omega_i(\gamma, y) \leq a(t, \|\gamma\|) \text{ reduces to}$$

$$\sqrt{\gamma^2 + y^2} \leq \gamma^2 + y^2 \leq 2(\sqrt{\gamma^2 + y^2})^2$$

with the proviso that $b(\wp) = \wp$, and $a(\wp) = 2\wp^2$.

Furthermore, we deduce that using equation (3.4) and $\Omega_1(t, \gamma_1, \gamma_2) = \gamma_1^2$

$${}^C D_+^\beta \Omega(t, \gamma_1, \gamma_2) = \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \Omega(t, \gamma_1, \gamma_2) - \Omega(t_0, \gamma_0) - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^{r+1} C_r [\Omega(t-rh, \gamma - h^\beta f_i(t, \gamma_1, \gamma_2)) - \Omega(t_0, \gamma_0)] \right\}, t \geq t_0$$

$${}^C D_+^\beta \Omega_1(t, \gamma_1) = \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \gamma_1^2 - \gamma_{10}^2 + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) [\Omega(t-rh, \gamma - h^\beta f_1(t, \gamma_1)) - \gamma_{10}^2] \right\}, t \geq t_0$$

$$\begin{aligned}
{}^C D_+^\beta \Omega_1(t, \gamma_1) &= \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \gamma_1^2 - \gamma_{10}^2 + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) [(\gamma_1 - h^\beta f_1(t, \gamma_1))^2 - \gamma_{10}^2] \right\}, t \geq t_0 \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \gamma_1^2 - \gamma_{10}^2 + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) [\gamma_1^2 - 2\gamma_1 h^\beta f_1(t, \gamma_1) + h^{2\beta} f_1^2(t, \gamma_1) - \gamma_{10}^2] \right\}, t \geq t_0 \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \gamma_1^2 + \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) \gamma_1^2 - \gamma_{10}^2 - \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) \gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) h^\beta f_1(t, \gamma_1) \right\}, t \geq t_0 \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \left\{ \sum_{r=0}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) \gamma_1^2 - \sum_{r=0}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) \gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) h^\beta f_1(t, \gamma_1) \right\}, t \geq t_0 \quad (6.2)
\end{aligned}$$

Recall from equations (3.7) and (3.8) in [1] that

$$\limsup_{h \rightarrow 0^+} \frac{1}{h^\beta} \sum_{r=0}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) = \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} \text{ and } \lim_{h \rightarrow 0^+} \sum_{r=0}^{\lfloor \frac{t-t_0}{h} \rfloor} (-1)^r ({}^\beta C_r) = -1 \quad (6.3)$$

and substituting (6.3) into (6.2), we have,

$$\begin{aligned}
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} - \frac{\gamma_{10}^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1 f_1(t, \gamma_1) \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1 f_1(t, \gamma_1) \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1 \left(-15\gamma_1 - \frac{\gamma_2^2 \operatorname{cosec} \gamma_1}{2\gamma_1} + \gamma_1 \sin \gamma_2 + \frac{3\gamma_2^2 \sin \gamma_1}{\gamma_1} \right) \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} - 30\gamma_1^2 - \gamma_2^2 \operatorname{cosec} \gamma_1 + 2\gamma_1^2 \sin \gamma_2 + 6\gamma_2^2 \sin \gamma_1 \\
\text{As } t \rightarrow \infty, \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} &\rightarrow 0, \text{ so we now have that} \\
{}^C D_+^\beta \Omega_1(t, \gamma_1) &\leq -30\gamma_1^2 - \gamma_2^2 \operatorname{cosec} \gamma_1 + 2\gamma_1^2 \sin \gamma_2 + 6\gamma_2^2 \sin \gamma_1 \\
&= 2\gamma_1^2 (-15 + \sin \gamma_2) + \gamma_2^2 (6 \sin \gamma_1 - \operatorname{cosec} \gamma_1)
\end{aligned}$$

$$\leq 2\gamma_1^2(-15 + |\sin \gamma_2|) + \gamma_2^2(6|\sin \gamma_1| - \frac{1}{|\sin \gamma_1|})$$

$$\leq 2\gamma_1^2(-15 + 1) + \gamma_2^2(6 - 1)$$

$$= \gamma_1^2(-14) + \gamma_2^2(5)$$

Therefore,

$${}^c D_+^\beta \Omega_1(t, \gamma_1) \leq -14\Omega_1 + 5\Omega_2 \quad (6.4)$$

Also for $\gamma_0 \in S_\psi$, for $t = t_k$, $k = 1, 2, \dots$ we have, $\Omega_1(t, \gamma(t) + \xi_k) = |\xi_k + \gamma(t)| \leq \Omega_1(t, \gamma(t))$

Similarly, we compute for the Dini derivative for $\Omega_2(t, \gamma_2) = \gamma_2^2$ and follow through the same argument by substituting for $f_2(t, \gamma_2) = {}^c D^\beta \gamma_2(t) = \frac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin \gamma_1 - 5\gamma_2 \operatorname{cosec} \gamma_1 - \gamma_1^2 \cos \gamma_2$ in equation (3.4) to have that,

$${}^c D_+^\beta \Omega_2(t, \gamma_2) \leq 4\Omega_1 - 12\Omega_2 \quad (6.5)$$

Also for $\gamma_0 \in S_\psi$, for $t = t_k$, $k = 1, 2, \dots$ we have, $\Omega_2(t, \gamma(t) + \tau_k) = |\tau_k + \gamma(t)| \leq \Omega_2(t, \gamma(t))$

By combining equations (6.4) and (6.5), we have

$${}^c D_+^\beta \Omega \leq \begin{pmatrix} -14 & 5 \\ 4 & -12 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = g(t, \Omega) \quad (6.6)$$

Now, consider the comparison system ${}^c D^\beta u = g(t, \Omega) = Au$, where $A = \begin{pmatrix} -14 & 5 \\ 4 & -12 \end{pmatrix}$.

The vectorial inequality (6.6) and all other conditions of Theorem 5.2 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (6.1) is uniformly practically stable. Therefore, the trivial solution $\gamma_0 = 0$ of the IFrDE (6.1) is uniformly practically stable.

7 Conclusion

In this paper, we examined the uniform practical stability of nonlinear impulsive Caputo fractional differential equations with fixed moment of impulse, using an auxiliary Lyapunov functions which are analogues of vector Lyapunov functions, and is generalized by a class of

piecewise continuous Lyapunov functions. By splitting the vector Lyapunov functions $V(t, \gamma)$ into several components of $V_1, V_2, \dots$, each of the state variable or solution state $x_1, x_2, \dots, x_m$ can be inserted into each of the components rather than into one Lyapunov function, so that the arguments to obtain the desired practical stability result becomes less complicated, and the imposed conditions would be less restrictive. Together with the comparison results, sufficient conditions for the uniform practical stability of the impulsive Caputo fractional order system (3.3) is established with an illustrative example.

Conflicts of Interest: The authors declare that there is no conflict of interest.

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